

# Strings next to and inside black holes

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## Abstract

The string equations of motion and constraints are solved near the horizon and near the singularity of a Schwarzschild black hole.

In a conformal gauge such that  $\tau = 0$  ( $\tau$  = worldsheet time coordinate) corresponds to the horizon ( $r = 1$ ) or to the black hole singularity ( $r = 0$ ), the string coordinates express in power series in  $\tau$  near the horizon and in power series in  $\tau^{1/5}$  around  $r = 0$ .

We compute the string invariant size and the string energy-momentum tensor. Near the horizon both are finite and analytic. Near the black hole singularity, the string size, the string energy and the transverse pressures (in the angular directions) tend to infinity as  $r^{-1}$ . To leading order near  $r = 0$ , the string behaves as two dimensional radiation. This two spatial dimensions are describing the  $S^2$  sphere in the Schwarzschild manifold.

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# I. INTRODUCTION AND MOTIVATIONS

The systematic investigation of strings in curved spacetimes started in [?,?] has uncovered a variety of new physical phenomena (see [?] for a review). These results are relevant both for fundamental (quantum) strings and for cosmic strings, which behave in an essentially classical way [?].

Black-hole spacetimes are probably the most relevant backgrounds to study strings propagating into a spacetime singularity (see [?] for plane wave null singularities). Strings in Schwarzschild black holes have been previously investigated in refs. [?,?,?,?]. Different kinds of symmetric Ansätze that **separate** the variables  $\sigma$  and  $\tau$  have been proposed. In this way the string evolution equations (which are non-linear partial differential equations) become ordinary differential equations.

In ref. [?] stationary solutions are obtained with the spatial variables depending only on  $\sigma$  and the temporal variable set as  $t = \tau$ . In this way infinitely long string solutions outside the horizon were constructed. String solutions inside the horizon are obtained in ref. [?] by exchanging  $\sigma \leftrightarrow \tau$  in the previous Ansatz. These string are also infinitely long. All solutions are expressed in terms of elliptic functions.

Ring solutions are obtained in ref. [?] through the Ansatz  $\phi = n\sigma$  and assuming the other coordinates to depend only on  $\tau$ . These solutions describe strings that can propagate from infinity. As they approach the black hole, they may fall or not into the singularity depending on the initial conditions. A special case describes strings oscillating inside the horizon [?,?]. This special solution expresses in terms of elementary functions.

We construct in this paper **generic** solutions of the string equations of motion and constraints in Schwarzschild black holes. We do that near the horizon and near the black hole singularity. We consider here closed strings. The generalization to open strings is straightforward.

Through appropriate conformal transformations on the world-sheet, we map in the generic case the intersection of the string world-sheet with the horizon into the line  $\tau = 0$ . A similar transformation can be performed for the black hole singularity  $r = 0$ . We then study the string equations of motion and constraints by expanding in  $\tau$  around  $\tau = 0$ . Near the black hole singularity, it turns out that the string solutions possess a series expansion in integer powers of  $\tau^{1/5}$ . Around the horizon, strings solutions are analytic in  $\tau$  and only integer powers of  $\tau$  appear.

Calling upon the conformal freedom, our solutions depend on four arbitrary functions of  $\sigma$ . These functions can be expressed in terms of the initial data. We thus see that only the transverse coordinates are physical degrees of freedom.

In Kruskal-Szekeres coordinates  $u, v, \theta, \phi$  [see eq.(??)], a generic string solution behaves near the black hole singularity as follows:

$$\begin{aligned} u(\sigma, \tau) &= e^{a(\sigma)} \left\{ 1 - \frac{1}{4} \gamma(\sigma)^4 \tau^{4/5} \left[ 1 + O(\tau^{2/5}) \right] \right. \\ &\quad \left. - \gamma(\sigma)^6 \frac{f'(\sigma)\nu(\sigma) \sin^2 g(\sigma) + \mu(\sigma)}{28 a'(\sigma)} \tau^{7/5} \left[ 1 + O(\tau^{2/5}) \right] \right\} , \\ v(\sigma, \tau) &= e^{-a(\sigma)} \left\{ 1 - \frac{1}{4} \gamma(\sigma)^4 \tau^{4/5} \left[ 1 + O(\tau^{2/5}) \right] \right. \end{aligned}$$

$$\begin{aligned}
& + \gamma(\sigma)^6 \frac{f'(\sigma)\nu(\sigma)\sin^2 g(\sigma) + \mu(\sigma)}{28 a'(\sigma)} \tau^{7/5} \left[1 + O(\tau^{2/5})\right] \Big\} , \\
\theta(\sigma, \tau) &= g(\sigma) + \tau^{1/5} \mu(\sigma) + O(\tau^{3/5}), \\
\phi(\sigma, \tau) &= f(\sigma) + \tau^{1/5} \nu(\sigma) + O(\tau^{3/5}).
\end{aligned} \tag{1}$$

where

$$[2\gamma(\sigma)]^2 = \mu(\sigma)^2 + \nu(\sigma)^2 \sin^2 g(\sigma) . \tag{2}$$

In addition, the (time-like) coordinate  $r$  vanishes as,

$$r(\sigma, \tau) = \gamma(\sigma)^2 \tau^{2/5} + O(\tau^{4/5}) . \tag{3}$$

Notice that the angular coordinates  $\theta, \phi$  vary with  $\tau$  near  $\tau = 0, r = 0$  faster than the  $u, v$  or  $r$  coordinates.

The string behaviour near the horizon is analytic in  $\tau$ . It is given in sec.III, eqs.(??-??).

We then compute the string size and energy-momentum near the black hole singularity. The invariant size  $S$  [?] grows as  $r^{-1}$  for  $r \rightarrow 0$ ,

$$S = \frac{4 a'(\sigma)^2}{r} + O(1) \tag{4}$$

The string infinitely stretches when it falls into the  $r = 0$  singularity. This is due to the infinitely growing gravitational forces that act then on the string. The string stretching near  $r = 0$  was observed in ref. [?] using perturbative methods.

The string energy diverges also as  $r^{-1}$  for  $r \rightarrow 0$ ,

$$E = \frac{1}{2\pi\alpha'} \frac{2}{5r} \int_0^{2\pi} d\sigma |\gamma(\sigma)|^5 + O(1) \rightarrow +\infty , \tag{5}$$

where  $2\pi\alpha'$  stands for the inverse string tension.

The pressure in the angular directions  $\theta, \phi$  tends to infinity with the same rate:

$$\begin{aligned}
P_\phi &= \frac{1}{2\pi\alpha'} \frac{1}{10r} \int_0^{2\pi} d\sigma \nu(\sigma)^2 \sin^2 g(\sigma) |\gamma(\sigma)|^3 \rightarrow +\infty , \\
P_\theta &= \frac{1}{2\pi\alpha'} \frac{1}{10r} \int_0^{2\pi} d\sigma \mu(\sigma)^2 |\gamma(\sigma)|^3 \rightarrow +\infty .
\end{aligned} \tag{6}$$

Thus, to leading order,

$$E = P_\theta + P_\phi \quad \text{for } r \rightarrow 0 .$$

which is the behaviour of a two-dimensional ultrarelativistic gas. This relation means that the string behaves to leading order as **two**-dimensional massless particles. This is the so-called dual to unstable behaviour [?,?] (here for two spatial dimension).

As is well known, the Schwarzschild manifold has a  $M \otimes S^2$  structure, where  $M$  stands for the part described by the  $u, v$  coordinates. The present results indicate that test strings near the black hole singularity behave as massless particles propagating in the compact  $S^2$  manifold.

Near the horizon both the string size and the energy-momentum tensor are finite and analytic in Kruskal-Szekeres coordinates. The trace of the energy-momentum tensor is positive at the horizon.

It must be noticed that the resolution method used here for strings in black hole spacetimes is analogous the expansions for  $\tau \rightarrow 0$  used in ref. [?,?] for strings in cosmological spacetimes.

For strings at large distances from the black hole where the gravitational field is weak, the string propagation can be solved by perturbing the Minkowski solutions. Actually, one can take as zeroth order solution the center of mass motion,  $q^A + 2p^A \alpha' \tau$  as it has been done in ref. [?].

In summary, the exact string behaviour near the black hole singularity and near the black hole horizon is presented in this paper.

## II. EQUATIONS OF MOTION.

The Schwarzschild metric in Schwarzschild coordinates  $(t, r, \theta, \phi)$  takes the following form:

$$ds^2 = - \left(1 - \frac{1}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{1}{r}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (7)$$

where we choose units where the Schwarzschild radius  $R_s = 2m = 1$ .

Since we are interested in the whole Schwarzschild manifold and not just in the external part  $r > 1$  where the static Schwarzschild coordinates are appropriate, we consider the Kruskal-Szekeres coordinates  $(u, v, \theta, \phi)$  defined by

$$u = t_K - r_K \equiv \sqrt{1 - r} e^{(r-t)/2}, \quad v = t_K + r_K \equiv \sqrt{1 - r} e^{(r+t)/2}. \quad (8)$$

for  $v \geq 0, u \geq 0$  and by

$$u = t_K - r_K \equiv -\sqrt{r - 1} e^{(r-t)/2}, \quad v = t_K + r_K \equiv \sqrt{r - 1} e^{(r+t)/2}. \quad (9)$$

for  $v \geq 0, u \leq 0$ . For  $v \leq 0$  one just flips the sign of  $v$  in eq.(?) or (??) [?].

The coordinate  $t_K$  is a time-like coordinate, and  $r_K$  spacelike. In Kruskal-Szekeres coordinates the Schwarzschild metric takes the form,

$$ds^2 = -\frac{4}{r} e^{-r} du dv + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (10)$$

$r$  is a function of the product  $uv$  defined by the inverse function of

$$uv = [1 - r] e^r.$$

for  $uv \geq 0$ . The metric in such coordinates is regular everywhere except at its singularity,  $r = 0$ .

The string equations of motion in Schwarzschild coordinates and in the conformal gauge, are

$$\begin{aligned}
& r_\sigma t_\sigma - r_\tau t_\tau + r(r-1)(t_{\sigma\sigma} - t_{\tau\tau}) = 0, \\
& \frac{2r}{1-r}(r_{\tau\tau} - r_{\sigma\sigma}) - \frac{1}{r^2}(t_\tau^2 - t_\sigma^2) + 2r(\theta_\tau^2 - \theta_\sigma^2) + \\
& 2r \sin^2 \theta (\phi_\tau^2 - \phi_\sigma^2) + \frac{1}{(r-1)^2}(r_\tau^2 - r_\sigma^2) = 0.
\end{aligned} \tag{11}$$

$$\begin{aligned}
& r \sin \theta (\phi_{\tau\tau} - \phi_{\sigma\sigma}) + 2r \cos \theta (\phi_\tau \theta_\tau - \phi_\sigma \theta_\sigma) + 2 \sin \theta (r_\tau \phi_\tau - r_\sigma \phi_\sigma) = 0, \\
& r (\theta_{\tau\tau} - \theta_{\sigma\sigma}) + 2(r_\tau \theta_\tau - r_\sigma \theta_\sigma) - r \sin \theta \cos \theta (\phi_\tau^2 - \phi_\sigma^2) = 0.
\end{aligned} \tag{12}$$

The constraints in Schwarzschild coordinates are

$$\begin{aligned}
& \frac{1-r}{r}(t_\sigma^2 + t_\tau^2) + \frac{r}{r-1}(r_\tau^2 + r_\sigma^2) + r^2 [\theta_\tau^2 + \theta_\sigma^2 + \sin^2 \theta (\phi_\tau^2 + \phi_\sigma^2)] = 0, \\
& \frac{1-r}{r}t_\tau t_\sigma + \frac{r}{r-1}r_\tau r_\sigma + r^2 (\theta_\tau \theta_\sigma + \sin^2 \theta \phi_\tau \phi_\sigma) = 0.
\end{aligned} \tag{13}$$

The string equations of motion in Kruskal-Szekeres coordinates take the form (always in the conformal gauge),

$$\begin{aligned}
& u_{\tau\tau} - u_{\sigma\sigma} + \frac{1}{r} \left(1 + \frac{1}{r}\right) e^{-r} v [(u_\tau)^2 - (u_\sigma)^2] - \frac{r u}{2} [\theta_\tau^2 - \theta_\sigma^2 + \sin^2 \theta (\phi_\tau^2 - \phi_\sigma^2)] = 0 \\
& v_{\tau\tau} - v_{\sigma\sigma} + \frac{1}{r} \left(1 + \frac{1}{r}\right) e^{-r} u [(v_\tau)^2 - (v_\sigma)^2] - \frac{r v}{2} [\theta_\tau^2 - \theta_\sigma^2 + \sin^2 \theta (\phi_\tau^2 - \phi_\sigma^2)] = 0,
\end{aligned} \tag{14}$$

plus eqs.(??) for the angular coordinates.

The constraints in Kruskal-Szekeres coordinates are

$$\begin{aligned}
& -\frac{4}{r} e^{-r} (u_\sigma v_\sigma + u_\tau v_\tau) + r^2 [\theta_\tau^2 + \theta_\sigma^2 + \sin^2 \theta (\phi_\tau^2 + \phi_\sigma^2)] = 0, \\
& -\frac{4}{r} e^{-r} (u_\tau v_\sigma + u_\sigma v_\tau) + r^2 (\theta_\tau \theta_\sigma + \sin^2 \theta \phi_\tau \phi_\sigma) = 0.
\end{aligned} \tag{15}$$

Notice that both the equations of motion and constraints are invariant under the exchange  $u \leftrightarrow v$ .

Also notice that the equations of motion and constraints in Kruskal-Szekeres coordinates are regular everywhere except at the singularity  $r = 0$ .

We shall consider closed strings where the string coordinates must be periodic functions of  $\sigma$ :

$$u(\sigma + 2\pi, \tau) = u(\sigma, \tau), \quad v(\sigma + 2\pi, \tau) = v(\sigma, \tau). \tag{16}$$

Therefore, the angular coordinates  $\theta, \phi$  may be just quasiperiodic functions of  $\sigma$ :

$$\theta(\sigma + 2\pi, \tau) = \theta(\sigma, \tau) + \text{mod } 2\pi, \quad \phi(\sigma + 2\pi, \tau) = \phi(\sigma, \tau) + 2n\pi, \tag{17}$$

where  $n$  is an integer.

### III. STRINGS ACROSS THE HORIZON

Let us study the string behaviour in the region near the horizon. The Kruskal-Szekeres coordinates are the appropriate ones here. The horizon is characterized by  $uv = 0$ . Let us consider a string crossing the right part of it, that is  $u = 0, v > 0$ .

We can write the curve describing the intersection of the horizon with the world-sheet as

$$x_+ = \chi(x_-), \quad (18)$$

whenever this intersection is nondegenerate. (Here  $x_{\pm} \equiv \tau \pm \sigma$ ).

Upon a conformal transformation,

$$x_+ \rightarrow x'_+ = f(x_+) \quad , \quad x_- \rightarrow x'_- = g(x_-), \quad (19)$$

we can map the curve (??) into  $\tau' = 0$  by an appropriate choice of  $f$  and  $g$ . For example, we can choose

$$f(x_+) = x_+ \quad , \quad g(x_-) = -\chi(x_-).$$

This defines our choice of gauge. From now on, we rename  $\tau'$  and  $\sigma'$  by  $\tau$  and  $\sigma$ , respectively. Notice that this choice does not completely fix the gauge. We can still perform transformations that leave the line  $\tau = 0$  unchanged. This is the case for the following class of conformal mappings

$$x_+ \rightarrow x'_+ = \varphi(x_+) \quad , \quad x_- \rightarrow x'_- = -\varphi(-x_-), \quad (20)$$

where  $\varphi(x)$  is an arbitrary function. Eq. (??) can be written as,

$$\begin{aligned} \tau' &= \frac{1}{2} [\varphi(\tau + \sigma) - \varphi(\sigma - \tau)] = \tau \varphi'(\sigma) + \frac{1}{6} \tau^3 \varphi'''(\sigma) + O(\tau^4) \\ \sigma' &= \frac{1}{2} [\varphi(\tau + \sigma) + \varphi(\sigma - \tau)] = \varphi(\sigma) + \frac{1}{2} \tau^2 \varphi''(\sigma) + O(\tau^4) \end{aligned} \quad (21)$$

The transformations (??) represent a diagonal subgroup of the set of left-right conformal transformations (??).

#### A. Equatorial Solutions

Let us first investigate strings on the plane  $\theta = \pi/2$ . This restriction is compatible with the equations of motion and constraints and it means that the string moves in an equatorial plane.

We can then propose the following expansion near  $v = 0$ ,

$$\begin{aligned} v(\sigma, \tau) &= e \left[ \tau c_1(\sigma) + \tau^2 c_2(\sigma) + O(\tau^3) \right] \\ u(\sigma, \tau) &= b_0(\sigma) + \tau b_1(\sigma) + \tau^2 b_2(\sigma) + O(\tau^3) \\ \phi(\sigma, \tau) &= p_0(\sigma) + \tau p_1(\sigma) + \tau^2 p_2(\sigma) + O(\tau^3) , \end{aligned} \quad (22)$$

where  $e = 2.718281828459 \dots$ . Inserting eqs.(??) in eqs.(??,??) and constraints (??) yields

$$\begin{aligned} c_2(\sigma) &= -b_0(\sigma) c_1(\sigma)^2, \\ b_1(\sigma) &= \frac{c_1(\sigma) [b'_0(\sigma)]^2}{[p'_0(\sigma)]^2} + \frac{[p'_0(\sigma)]^2}{4 c_1(\sigma)}, \\ p_1(\sigma) &= \frac{2c_1(\sigma) b'_0(\sigma)}{p'_0(\sigma)}. \end{aligned} \quad (23)$$

In addition  $r(\sigma, \tau)$  is given by the expansion

$$r(\sigma, \tau) = 1 - b_0(\sigma) c_1(\sigma) \tau - \left\{ \frac{[c_1(\sigma) b'_0(\sigma)]^2}{[p'_0(\sigma)]^2} + \frac{1}{4} [p'_0(\sigma)]^2 \right\} \tau^2 + O(\tau^3). \quad (24)$$

The expansion can be pushed to arbitrary high orders in  $\tau$  yielding the series coefficients in terms of the arbitrary functions  $b_0(\sigma)$ ,  $c_1(\sigma)$  and  $p_0(\sigma)$ . We have easily generated a number of terms with the help of Mathematica. The number of arbitrary functions can be reduced to two using the conformal freedom (??).

The boundary conditions (??-??) require here:

$$b_0(\sigma + 2\pi) = b_0(\sigma), \quad c_1(\sigma + 2\pi) = c_1(\sigma), \quad p_0(\sigma + 2\pi) = p_0(\sigma) + 2n\pi. \quad (25)$$

## B. General String Solutions

Let us consider now a general string solution near the horizon  $r = 1$ . We keep working in a gauge such that  $\tau = 0$  corresponds to  $r = 1$ .

Eqs. (??) are then supplemented by

$$\theta(\sigma, \tau) = t_0(\sigma) + t_1(\sigma) \tau + t_2(\sigma) \tau^2 + O(\tau^3). \quad (26)$$

Inserting eqs.(??, ??) into eqs.(??,??) and constraints (??) yields

$$\begin{aligned} u(\sigma, \tau) &= b_0 + \\ &+ \frac{\tau}{4c_1} \left[ (p_1 + p'_0)^2 \sin^2 t_0 + (t_1 + t'_0)^2 \right] + \\ &+ \frac{\tau^2}{4} \left[ b_0 (p_1^2 - p_0'^2) \sin^2 t_0 + b_0 (t_1^2 - t_0'^2) + 2b_0'' \right] + O(\tau^3), \\ v(\sigma, \tau) &= e \left\{ c_1 \tau - b_0 c_1^2 \tau^2 \right. \\ &+ \frac{\tau^3}{12} \left[ 6b_0^2 c_1^3 + 4c_1^2 b'_0 + 2c_1'' - c_1 p'_0 (p_1 + p'_0) \right. \\ &\quad \left. \left. - 2c_1 t'_0 (t'_0 + t_1) + \cos(2t_0) c_1 p'_0 (p_1 + p'_0) \right] + O(\tau^4) \right\}, \\ r(\sigma, \tau) &= 1 - b_0 c_1 \tau + \end{aligned}$$

$$\begin{aligned}
& -\frac{\tau^2}{4} \left[ \sin^2 t_0 (p'_0 + p_1)^2 + (t'_0 + t_1)^2 c_1 b'_0 \right] + O(\tau^3) , \\
\phi(\sigma, \tau) &= p_0 + p_1 \tau + \left[ b_0 c_1 p_1 - (p_1 t_1 - p'_0 t'_0) \cot t_0 + \frac{p''_0}{2} \right] \tau^2 + O(\tau^3) , \\
\theta(\sigma, \tau) &= t_0 + t_1 \tau + \left[ \frac{\sin(2 t_0)}{4} (p_1^2 - p'^2_0) + b_0 c_1 t_1 + \frac{t''_0}{2} \right] \tau^2 + O(\tau^3) .
\end{aligned} \tag{27}$$

Here,

$$p_1(\sigma) = \frac{2 c_1 b'_0 - t_1 t'_0}{p'_0 \sin^2 t_0} . \tag{28}$$

We have here five arbitrary functions  $b_0(\sigma), c_1(\sigma), p_0(\sigma), t_0(\sigma)$  and  $t_1(\sigma)$ . This number can be reduced to four using the diagonal conformal transformations (??).

#### IV. STRINGS NEAR THE SINGULARITY $R = 0$

Let us consider the solution of eqs.(??,??) and constraints (??), near  $r = 0$ . That is to say, near  $uv = 1$ .

For a generic world-sheet, we choose the gauge such that  $\tau = 0$  corresponds to the string at the singularity  $uv = 1$ . This can be achieved as shown in sec. III for the horizon.

Notice that we use in sec. III and IV the same symbols  $(\sigma, \tau)$  for different world-sheet coordinates. In sec. III,  $\tau = 0$  corresponds to  $r = 1$ , whereas in sec. IV,  $\tau = 0$  corresponds to  $r = 0$ .

##### A. Equatorial Solutions

Let us begin to investigate strings on the plane  $\theta = \pi/2$ . This restriction is compatible with the equations of motion and constraints and it means that the string moves in an equatorial plane.

Near the singularity  $uv = 1$ , we propose for  $\tau \rightarrow 0$  the Ansatz

$$\begin{aligned}
u(\sigma, \tau) &= e^{a(\sigma)} [1 - \tau^\alpha \beta(\sigma) + \dots] \\
v(\sigma, \tau) &= e^{-a(\sigma)} [1 - \tau^{\alpha'} \hat{\beta}(\sigma) + \dots] \\
\phi(\sigma, \tau) &= f(\sigma) + 2\gamma(\sigma) \tau^\lambda + \dots .
\end{aligned} \tag{29}$$

[The factor 2 is there just for future convenience].

Inserting eqs.(??) in eqs.(??,??) and constraints (??) yields

$$\begin{aligned}
\alpha = \alpha' &= 4/5 \quad , \quad \lambda = 1/5, \\
\hat{\beta}(\sigma) &= \beta(\sigma) \quad , \quad \beta(\sigma) = \frac{1}{4} \gamma(\sigma)^4 .
\end{aligned} \tag{30}$$



The functions  $a(\sigma)$ ,  $f(\sigma)$  and  $\gamma(\sigma)$  are arbitrary.

The coordinate  $r$  then vanishes as

$$r(\sigma, \tau) = \gamma(\sigma)^2 \tau^{2/5} + \dots \quad (31)$$

We have pushed the resolution of eqs.(??,??) and constraints (??) to higher orders with the help of Mathematica. We find that the corrections to the leading behaviour appear as integer powers of  $\tau^{2/5}$ . That is, terms in  $\tau^{6/5}, \tau^{8/5}, \tau^{10/5}, \dots$  in  $u$  and in  $v$ , and terms in  $\tau^{3/5}, \tau^{5/5}, \tau^{7/5}, \dots$  in  $\phi$ . The subdominant contributions start with the order  $\tau^{7/5}$  in  $u$  and in  $v$ . The piece in  $\phi$  of order  $\tau^0$  is directly connected with the subdominant contributions in  $u$  and in  $v$ .

We find,

$$\begin{aligned} u(\sigma, \tau) &= e^{a(\sigma)} \left\{ 1 - \frac{1}{4} \gamma(\sigma)^4 \tau^{4/5} - \frac{1}{21} \left( 2\gamma(\sigma)^6 + \frac{75}{2} \frac{a'(\sigma)^2}{\gamma(\sigma)^4} \right) \tau^{6/5} \left[ 1 + O(\tau^{2/5}) \right] \right. \\ &\quad \left. - \frac{\gamma(\sigma)^7 f'(\sigma)}{14 a'(\sigma)} \tau^{7/5} \left[ 1 + O(\tau^{2/5}) \right] \right\}, \\ v(\sigma, \tau) &= e^{-a(\sigma)} \left\{ 1 - \frac{1}{4} \gamma(\sigma)^4 \tau^{4/5} - \frac{1}{21} \left( 2\gamma(\sigma)^6 + \frac{75}{2} \frac{a'(\sigma)^2}{\gamma(\sigma)^4} \right) \tau^{6/5} \left[ 1 + O(\tau^{2/5}) \right] \right. \\ &\quad \left. + \frac{\gamma(\sigma)^7 f'(\sigma)}{14 a'(\sigma)} \tau^{7/5} \left[ 1 + O(\tau^{2/5}) \right] \right\}, \\ \phi(\sigma, \tau) &= f(\sigma) + \frac{5}{21} \left( \frac{4 f'(\sigma) a''(\sigma)}{7 a'(\sigma)} + \frac{4 f'(\sigma) \gamma'(\sigma)}{3 \gamma(\sigma)} + \frac{53}{42} f''(\sigma) \right) \tau^2 + O(\tau^{12/5}) \\ &\quad + 2 \gamma(\sigma) \tau^{1/5} \left\{ 1 + \frac{2}{21 \gamma(\sigma)^8} \left[ \gamma(\sigma)^{10} - 25 a'(\sigma)^2 \right] \tau^{2/5} + O(\tau^{4/5}) \right\}, \end{aligned} \quad (32)$$

and

$$\begin{aligned} r(\sigma, \tau) &= \gamma(\sigma)^2 \tau^{2/5} \left[ 1 + \frac{\tau^{2/5}}{7} \left( -\gamma(\sigma)^2 + \frac{25 a'(\sigma)^2}{\gamma(\sigma)^6} \right) + O(\tau^{4/5}) \right] \\ &\quad + \frac{25}{441} \tau^{11/5} \left[ 35 \gamma'(\sigma) f'(\sigma) - \frac{6 f'(\sigma) \gamma(\sigma) a''(\sigma)}{a'(\sigma)} - \gamma(\sigma) f''(\sigma) + O(\tau^{2/5}) \right]. \end{aligned} \quad (33)$$

The functions  $f(\sigma)$ ,  $a(\sigma)$  and  $\gamma(\sigma)$  are arbitrary and can be expressed in terms of the initial data. In the Appendix we give  $u, v, \phi, r$  up to the order  $\tau^{16/5}$ .

Both the equations of motion and constraints are invariant under the exchange  $u \leftrightarrow v$  but not the boundary conditions at  $\tau = 0$ . They differ by  $a(\sigma) \leftrightarrow -a(\sigma)$  as we see from eqs.(??). Therefore one can obtain  $u(\sigma, \tau)$  from  $v(\sigma, \tau)$  and viceversa just by flipping the sign of  $a(\sigma)$ .

Notice that  $u/v$  is  $\tau$  independent up to order  $\tau^{7/5}$ . Since  $u/v = e^{-t}$ , this imply that the spatial coordinate  $t$  is only  $\sigma$ -dependent up to  $O(\tau^{7/5})$ . More precisely,

$$t(\sigma, \tau) = \log \frac{v}{u} = -2a(\sigma) + \frac{\gamma(\sigma)^7 f'(\sigma)}{7 a'(\sigma)} \tau^{7/5} + O(\tau^{9/5}). \quad (34)$$

In other words,  $t(\sigma, \tau)$  varies slower than the other coordinates  $\phi$  and  $r$  when the string approaches the black hole singularity ( $\tau \rightarrow 0$ ).

The boundary conditions (??-??) require here:

$$a(\sigma + 2\pi) = a(\sigma) , \quad \gamma(\sigma + 2\pi) = \gamma(\sigma) , \quad f(\sigma + 2\pi) = f(\sigma) + 2n\pi . \quad (35)$$

Recall that we can still perform diagonal conformal transformations (??). It is easy to check that eqs.(??) keep their form under the transformations (??) when we expand in powers of  $\tau$ .

We can use the arbitrary function  $\varphi(\sigma)$  to fix one function among  $f(\sigma)$ ,  $a(\sigma)$  and  $\gamma(\sigma)$ . For example, we can set to zero the  $O(\tau^2)$  part in  $\phi(\sigma, \tau)$  [see eq.(??)]. We are thus left with **two** arbitrary functions. They describe precisely the transverse degrees of freedom of the equatorial string.

## B. General String Solutions

Let us consider now a general string solution near the singularity  $r = 0$ . We keep working in a gauge such that  $\tau = 0$  corresponds to  $r = 0$ .

Eqs.(??) generalize as follows for the dominant order

$$\begin{aligned} u(\sigma, \tau) &= e^{a(\sigma)} \left[ 1 - \frac{1}{4} \gamma(\sigma)^4 \tau^{4/5} + \dots \right] , \\ v(\sigma, \tau) &= e^{-a(\sigma)} \left[ 1 - \frac{1}{4} \gamma(\sigma)^4 \tau^{4/5} + \dots \right] , \\ \theta(\sigma, \tau) &= g(\sigma) + \tau^{1/5} \mu(\sigma) + \dots , \\ \phi(\sigma, \tau) &= f(\sigma) + \tau^{1/5} \nu(\sigma) + \dots \end{aligned} \quad (36)$$

Inserting eqs.(??) in eqs.(??,??) and constraints (??) yields

$$[2\gamma(\sigma)]^2 = \mu(\sigma)^2 + \nu(\sigma)^2 \sin^2 g(\sigma) . \quad (37)$$

For  $g(\sigma) \equiv \pi/2$ ,  $\mu(\sigma)^2 \equiv 0$  we get back the previous equatorial solution.

We can also find the ring solution of ref. [?] setting  $f(\sigma) \equiv n\sigma$ ,  $a(\sigma) \equiv 0$ ,  $\mu(\sigma) = \text{cte}$ .  $g(\sigma) = \text{cte}$ . and  $\nu(\sigma) \equiv 0$ .

The coordinate  $r$  vanishes here as for the equatorial solution

$$r(\sigma, \tau) = \gamma(\sigma)^2 \tau^{2/5} + \dots . \quad (38)$$

The string solution is completely fixed once the functions  $f(\sigma)$ ,  $g(\sigma)$ ,  $a(\sigma)$ ,  $\mu(\sigma)$  and  $\nu(\sigma)$  are given. These five functions are arbitrary and can be expressed in terms of the initial data.

Notice that  $\phi$  and  $\theta$  approach their limiting values with the same exponent  $1/5$  in  $\tau$ .

As in the equatorial case, the corrections to the leading behaviour appear as positive integer powers of  $\tau^{2/5}$ . The subdominant leading power in  $u(\sigma, \tau)$  and  $v(\sigma, \tau)$  is again  $\tau^{7/5}$ . We find with the help of Mathematica,

$$u(\sigma, \tau) = e^{a(\sigma)} \left\{ 1 - \gamma(\sigma)^4 \tau^{4/5} \left[ 1 + O(\tau^{2/5}) \right] \right.$$

$$\begin{aligned}
& - \gamma(\sigma)^6 \frac{f'(\sigma)\nu(\sigma)\sin^2 g(\sigma) + \mu(\sigma)}{28 a'(\sigma)} \tau^{7/5} \left[1 + O(\tau^{2/5})\right] \Big\} , \\
v(\sigma, \tau) &= e^{-a(\sigma)} \left\{1 - \gamma(\sigma)^4 \tau^{4/5} \left[1 + O(\tau^{2/5})\right] \right. \\
& \left. + \gamma(\sigma)^6 \frac{f'(\sigma)\nu(\sigma)\sin^2 g(\sigma) + \mu(\sigma)}{28 a'(\sigma)} \tau^{7/5} \left[1 + O(\tau^{2/5})\right] \right\} . \tag{39}
\end{aligned}$$

Using the diagonal conformal transformation (??), we can fix one of the arbitrary functions among  $f(\sigma), g(\sigma), a(\sigma), \mu(\sigma)$  and  $\nu(\sigma)$  keeping in mind the periodic boundary conditions:

$$\begin{aligned}
a(\sigma + 2\pi) &= a(\sigma) , \quad \nu(\sigma + 2\pi) = \nu(\sigma) , \quad \mu(\sigma + 2\pi) = \mu(\sigma) , \\
f(\sigma + 2\pi) &= f(\sigma) + 2n\pi , \quad g(\sigma + 2\pi) = g(\sigma) \bmod 2\pi . \tag{40}
\end{aligned}$$

We are left with **four** arbitrary functions of  $\sigma$ . This is precisely the number of transverse string degrees of freedom.

## V. STRING ENERGY-MOMENTUM AND INVARIANT SIZE NEAR THE SINGULARITY AND THE NEAR THE HORIZON

The invariant string size  $ds^2$  is defined using the metric induced on the string world-sheet [?]:

$$ds^2 = G_{AB}(X) \dot{X}^A \dot{X}^B (d\tau^2 - d\sigma^2) . \tag{41}$$

We give the name *string size* to

$$S = -G_{AB}(X) \dot{X}^A \dot{X}^B = \frac{4}{r} e^{-r} \dot{u} \dot{v} - r^2 \dot{\theta}^2 - r^2 \dot{\phi}^2 \sin^2 \theta . \tag{42}$$

We find near the singularity at  $r = 0$  using eqs.(??-??)

$$\begin{aligned}
S &= \frac{4 a'(\sigma)^2}{\gamma(\sigma)^2} \tau^{-2/5} - \frac{4}{7} a'(\sigma)^2 \left(6 + 25 \frac{a'(\sigma)^2}{\gamma(\sigma)^7}\right) + O(\tau^{2/5}) \\
&= \frac{4 a'(\sigma)^2}{r} - \frac{4}{7} a'(\sigma)^2 \left(6 + 25 \frac{a'(\sigma)^2}{\gamma(\sigma)^7}\right) + O(r) . \tag{43}
\end{aligned}$$

The invariant string size tends then to infinite when the string falls into the  $r = 0$  singularity. This is due to the infinitely growing gravitational forces that act there on the string.

The string stretching near  $r = 0$  was observed in ref. [?] using perturbative methods.

We find near the horizon using eqs.(??)-(??) and (??),

$$S = [p'_0(\sigma)]^2 + O(\tau) = [p'_0(\sigma)]^2 + O(r - 1) . \tag{44}$$

As expected, we find a finite result since  $r = 1$  is a regular point of the geometry.

The spacetime string energy-momentum tensor in four spacetime dimensions is given by

$$\sqrt{-G} T^{AB}(X) = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \left( \dot{X}^A \dot{X}^B - X'^A X'^B \right) \delta^{(4)}(X - X(\sigma, \tau)) . \quad (45)$$

Notice that  $X$  in eq.(??) is just a spacetime point whereas  $X(\sigma, \tau)$  stands for the string dynamical variables. One sees from the Dirac delta in eq.(??) that  $T^{AB}(X)$  vanishes unless  $X$  is exactly on the string world-sheet. We shall not be interested in the detailed structure of the classical strings. It is more useful to integrate the energy-momentum tensor (??) on a volume that completely encloses the string as in refs. [?], [?]. We obtain in this way a density  $\Theta^{AB}$ .

Inside the horizon we can use  $t, \theta, \phi$  as spatial coordinates and  $r$  as a coordinate time. We find,

$$\sqrt{g_{rr}} \Theta^{AB}(r) = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \left( \dot{X}^A \dot{X}^B - X'^A X'^B \right) \delta(r - r(\tau, \sigma)). \quad (46)$$

where  $g_{rr} = r/(1 - r) > 0$ .

Let us first compute the trace of the energy-momentum tensor (??). We find

$$\sqrt{-G} T_A^A(X) = \frac{1}{\pi\alpha'} \int d\sigma d\tau G_{AB}(X) \dot{X}^A \dot{X}^B \delta^{(4)}(X - X(\sigma, \tau)) . \quad (47)$$

where we used the string constraints. Notice that the integrand is just the (minus) string size [eq.(??)].

We have for the black hole case:

$$G_{AB}(X) \dot{X}^A \dot{X}^B = -\frac{r\dot{r}^2}{1-r} + r^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 \sin^2 \theta + \frac{1-r}{r} \dot{t}^2 . \quad (48)$$

Using eqs.(??) and (??) for  $\tau \rightarrow 0$ , we find that each of the first three terms grows as  $\tau^{-4/5}$  whereas the last term vanishes as  $\tau^{2/5}$ . Moreover, the sum of the three terms  $O(\tau^{-4/5})$  identically vanishes thanks to eq.(??). This cancellation in the trace tells us that near  $r = 0$ , the dominant (and divergent) components  $T_r^r, T_\phi^\phi$  and  $T_\theta^\theta$  yield a zero trace. This means that the string behaves to leading order as **two**-dimensional massless particles. This is the so-called dual to unstable behaviour [?] (here for two spatial dimension).

For  $\tau \rightarrow 0$ ,  $r \rightarrow 0$  we can use in eq.(??) the dominant behaviours:

$$\begin{aligned} r(\sigma, \tau) &= \gamma(\sigma)^2 \tau^{2/5} + O(\tau^{4/5}), \\ \theta(\sigma, \tau) &= g(\sigma) + \mu(\sigma) \tau^{1/5} + O(\tau^{3/5}), \\ \phi(\sigma, \tau) &= f(\sigma) + \nu(\sigma) \tau^{1/5} + O(\tau^{3/5}), \\ t(\sigma, \tau) &= -2a(\sigma) + \gamma(\sigma)^6 \frac{f'(\sigma)\nu(\sigma) \sin^2 g(\sigma) + \mu(\sigma)}{14 a'(\sigma)} \tau^{7/5} + O(\tau^{9/5}) . \end{aligned} \quad (49)$$

We thus find for  $r \rightarrow 0$ ,

$$2\pi\alpha' \Theta^{rr}(r) = \frac{2}{5 r^2} \int_0^{2\pi} d\sigma |\gamma(\sigma)|^5 + O\left(\frac{1}{r}\right) \rightarrow +\infty ,$$

$$\begin{aligned}
2\pi\alpha' \Theta^{\phi\phi}(r) &= \frac{1}{10 r^3} \int_0^{2\pi} d\sigma \nu(\sigma)^2 |\gamma(\sigma)|^3 + O\left(\frac{1}{r^2}\right) \rightarrow +\infty, \\
2\pi\alpha' \Theta^{\theta\theta}(r) &= \frac{1}{10 r^3} \int_0^{2\pi} d\sigma \mu(\sigma)^2 |\gamma(\sigma)|^3 + O\left(\frac{1}{r^2}\right) \rightarrow +\infty, \\
2\pi\alpha' \Theta^{tt}(r) &= -10 r \int_0^{2\pi} d\sigma \frac{[a'(\sigma)]^2}{|\gamma(\sigma)|^5} + O(r^2) \rightarrow 0^- .
\end{aligned} \tag{50}$$

We can identify the string energy with the mixed component  $-\Theta_r^r$ . We define the mixed components  $\Theta_A^B(r)$  by integrating  $T_A^B(X)$  over the spatial volume.

This yields for  $r \rightarrow 0$ ,

$$E \equiv -\Theta_r^r = \frac{1}{2\pi\alpha'} \frac{2}{5 r} \int_0^{2\pi} d\sigma |\gamma(\sigma)|^5 + O(1) \rightarrow +\infty . \tag{51}$$

The transverse pressures are defined as the mixed components  $\Theta_\phi^\phi$  and  $\Theta_\theta^\theta$ . They diverge for  $r \rightarrow 0$  :

$$\begin{aligned}
P_\phi \equiv \Theta_\phi^\phi &= \frac{1}{2\pi\alpha'} \frac{2}{5 r} \int_0^{2\pi} d\sigma \nu(\sigma)^2 \sin^2 g(\sigma) |\gamma(\sigma)|^5 \rightarrow +\infty, \\
P_\theta \equiv \Theta_\theta^\theta &= \frac{1}{2\pi\alpha'} \frac{2}{5 r} \int_0^{2\pi} d\sigma \mu(\sigma)^2 |\gamma(\sigma)|^5 \rightarrow +\infty.
\end{aligned} \tag{52}$$

Thus, to leading order,

$$E = P_\theta + P_\phi \quad \text{for } r \rightarrow 0 .$$

exhibiting a two-dimensional ultrarelativistic gas behaviour. The tidal forces infinitely stretch the string near  $r = 0$  in effectively only two directions:  $\phi$  and  $\theta$ .

We find for the off-diagonal components,

$$\begin{aligned}
2\pi\alpha' \Theta^{tr}(r) &= \frac{r^{1/2}}{10} \int_0^{2\pi} d\sigma \frac{\nu(\sigma)}{a'(\sigma)} \gamma(\sigma)^4 \left[ f'(\sigma) \nu(\sigma) \sin^2 g(\sigma) + \mu(\sigma) \right] \rightarrow 0^+, \\
2\pi\alpha' \Theta^{t\theta}(r) &= \frac{2}{5} \int_0^{2\pi} d\sigma \frac{\mu(\sigma)}{a'(\sigma)} |\gamma(\sigma)|^3 \left[ f'(\sigma) \nu(\sigma) \sin^2 g(\sigma) + \mu(\sigma) \right] = O(1), \\
2\pi\alpha' \Theta^{t\phi}(r) &= \frac{2}{5} \int_0^{2\pi} d\sigma \frac{\nu(\sigma)}{a'(\sigma)} |\gamma(\sigma)|^3 \left[ f'(\sigma) \nu(\sigma) \sin^2 g(\sigma) + \mu(\sigma) \right] = O(1), \\
2\pi\alpha' \Theta^{r\phi}(r) &= \frac{1}{5 r^{5/2}} \int_0^{2\pi} d\sigma \nu(\sigma) \gamma(\sigma)^4 \rightarrow \infty, \\
2\pi\alpha' \Theta^{r\theta}(r) &= \frac{1}{5 r^{5/2}} \int_0^{2\pi} d\sigma \mu(\sigma) \gamma(\sigma)^4 \rightarrow \infty, \\
2\pi\alpha' \Theta^{\theta\phi}(r) &= \frac{1}{10 r^3} \int_0^{2\pi} d\sigma \mu(\sigma) \nu(\sigma) |\gamma(\sigma)|^3 \rightarrow \infty .
\end{aligned} \tag{53}$$

Notice that the invariant string size tends to infinity [see eq.(??)] with  $4a'(\sigma)^2$  as proportionality factor. Since  $-2a(\sigma)$  is the leading behaviour of  $t(\sigma, \tau)$ , this suggests us that the string stretches infinitely in the (spatial)  $t$  direction when  $r \rightarrow 0$ .

As a matter of fact, infinitely growing string sizes are not observed in cosmological spacetimes [?,?] for strings exhibiting radiation (dual to unstable) behaviour.

For particular string solutions the energy-momentum tensor and the string size can be less singular than in the generic case. For ring solutions [?],  $\mu(\sigma) = \mu = \text{constant}$ ,  $g(\sigma) = g = \text{constant}$ ,  $a(\sigma) = \nu(\sigma) = 0$ , there is no stretching and

$$S = r \sin g \rightarrow 0$$

$$E = P_\theta = \frac{1}{\alpha'} \frac{\mu^5}{80 r} \rightarrow +\infty \quad , \quad P_\phi = 0 \quad .$$

There is no string stretching but the string keeps exhibiting dual to unstable behaviour. This is due to the balance of the tidal forces thanks to the special symmetry of the solution. It behaves in this special case as **one**-dimensional massless particles for  $r \rightarrow 0$ .

As is easy to see, setting  $\mu(\sigma) = 0$ ,  $g(\sigma) = \pi/2$  all equatorial string solutions behave as **one**-dimensional massless particles for  $r \rightarrow 0$ .

It must be noticed that the resolution method used here for strings in black hole spacetimes is analogous the expansions for  $\tau \rightarrow 0$  used in ref. [?,?] for strings in cosmological spacetimes.

The string coordinates are regular near the horizon [see eqs. (??)]. Let us see that the energy-momentum tensor is also regular near the horizon. We use now Kruskal-Szekeres coordinates and define  $\Theta^{AB}(v)$  by integration over  $u, \theta$  and  $\phi$ . [For simplicity, we consider the equatorial solution]. We find for the energy  $\Theta^{t_K t_K}(v)$ ,

$$2\pi\alpha' \Theta^{t_K t_K}(v=0) = \frac{e^{-1/2}}{4} \int_0^{2\pi} d\sigma \frac{[b_1(\sigma) + e c_1(\sigma)]^2 - b_0'(\sigma)^2}{|c_1(\sigma)|} = O(1) \quad (54)$$

All other components of  $\Theta^{AB}(t_K)$  can be analogously computed and take finite values. We find for  $\Theta^{r_K r_K}(v)$ ,

$$2\pi\alpha' \Theta^{r_K r_K}(v=0) = \frac{e^{-1/2}}{4} \int_0^{2\pi} d\sigma \frac{[b_1(\sigma) - e c_1(\sigma)]^2 - b_0'(\sigma)^2}{|c_1(\sigma)|} = O(1) \quad (55)$$

For the energy-momentum trace we find

$$\pi\alpha' \Theta_A^A(t_K) = e^{1/2} \int_0^{2\pi} d\sigma [p_0(\sigma)]^2 = O(1) \quad (56)$$

We see that  $\Theta_A^A(t_K) > 0$  near the horizon.

## VI. STRINGS FAR AWAY FROM THE BLACK HOLE

At large distances from the black hole ( $r \gg 1$ ), the string equations of motion and constraints become those of Minkowski spacetime,

$$\begin{aligned} \partial_{-+} X^A(\sigma, \tau) &= 0 \quad , \quad 0 \leq A \leq 3, \\ \left[ \partial_{\pm} X^0(\sigma, \tau) \right]^2 - \sum_{j=1}^3 \left[ \partial_{\pm} X^j(\sigma, \tau) \right]^2 &= 0 \quad . \end{aligned} \quad (57)$$

The solution of eqs.(??) takes the customary form

$$X^A(\sigma, \tau) = q^A + 2p^A \alpha' \tau + i\sqrt{\alpha'} \sum_{n \neq 0} \frac{1}{n} \{ \alpha_n^A \exp[in(\sigma - \tau)] + \tilde{\alpha}_n^A \exp[-in(\sigma + \tau)] \} , \quad (58)$$

where  $q^A$  and  $p^A$  stand for the string center of mass position and momentum and  $\alpha_n^A$  and  $\tilde{\alpha}_n^A$  describe the right and left oscillator modes of the string, respectively. In order to have  $r \gg 1$ , we must assume  $\tau \gg 1$  so  $|q^A + 2p^A \alpha' \tau| \gg 1$ . Therefore, we can get the large distance approximation here as a systematic expansion in inverse powers of  $\tau$ . As zeroth order solution we can take:

$$q^A + 2p^A \alpha' \tau .$$

This is precisely the expansion constructed in ref. [?]. We refer to this article for strings propagating far away from the black hole. This regime is a weak field approximation. It can be considered as a newtonian approximation plus a post-newtonian one plus higher orders.

## VII. AN EXACT STRING SOLUTION

An exact solution of the string equations of motion and constraints [eqs. (??, ??-??)] is given by

$$\begin{aligned} u &= f(x_-) \quad , \quad v = g(x_+) \quad , \\ \phi &= 0 \quad , \quad \theta = \pi/2 . \end{aligned} \quad (59)$$

where  $f(x)$  and  $g(x)$  are arbitrary periodic functions of  $x$  with period  $2\pi$ . Even if the world-sheet reaches the  $r = 0$  singularity, it is not necessarily true that we can perform a conformal transformation on this solution such that  $\tau = 0$  corresponds to  $r = 0$ .

Let us see that this exact radial string solution has in fact zero energy-momentum. Inserting eqs.(??) in eq.(??), we see immediately that all components of  $T^{AB}(X)$  vanish except possibly for  $T^{UV}(X)$ . For this one,

$$\begin{aligned} 2\pi\alpha'\sqrt{-G} T^{UV}(X) &= \int dx_+ dx_- (\partial_+ U \partial_- V + \partial_- U \partial_+ V) \delta^{(4)}(X - X(\sigma, \tau)) \\ &= \int dx_+ dx_- f'(x_-) g'(x_+) \delta^{(4)}(X - X(\sigma, \tau)) \\ &= \delta(\phi) \delta(\theta - \pi/2) \sum_i \text{sign}[f'(x_-^i(u))] \sum_j \text{sign}[g'(x_+^j(v))] \end{aligned} \quad (60)$$

where  $x_-^i(u)$  and  $x_+^j(v)$  are the real roots of

$$u = f(x_-) \quad , \quad v = g(x_+) \quad , \quad (61)$$

for given  $u$  and  $v$ . Now, since  $f(x_-)$  and  $g(x_+)$  are periodic functions, eqs.(??) must have an even number of solutions. In addition,  $f'(x_-^i(u))$  and  $g'(x_+^j(v))$  will have positive and negative signs pairwise. Therefore, the sum over  $i, j$  will vanish in eq.(??):

$$T^{UV}(X) = 0 .$$

In summary, this exact radial solution does not carry energy-momentum. It is then a pure gauge object.

Solutions of this type have been previously discussed [?, ?, ?] and found to be either pure gauge or unphysical as they stand. In particular, the unphysical character of these solutions is suggested in ref. [?], insofar as they require (on top of the periodicity in  $\sigma$ ) that the physical time coordinate be an increasing function of  $\tau$ . This leads to folded string solutions, which are exact solutions parts of whose worldsheets are described by expressions (??), but with discontinuities in the first derivatives of the worldsheet to spacetime maps. In such a case, the stress-energy tensor is concentrated on the folds, as follows immediately from our computation above when applied to the specific situation at hand, thus leading to a system with closer resemblance to a grid than to a string worldsheet. That is to say, the lattice introduced for “mathematical convenience” only [?] has been mapped into a “grid” with physical significance in spacetime itself.

## APPENDIX A

In this appendix we give the explicit expansion of the string coordinates  $u(\sigma, \tau)$ ,  $v(\sigma, \tau)$ ,  $r(\sigma, \tau)$  and  $\phi(\sigma, \tau)$  around the  $r = 0$  singularity for the equatorial case. We choose a gauge on the world-sheet such that  $r = 0$  corresponds to  $\tau = 0$ .

$$\begin{aligned}
e^a u = & 1 - \frac{\gamma^4}{4} \tau^{4/5} - \\
& - \left( \frac{2\gamma^6}{21} + \frac{25a'^2}{14\gamma^4} \right) \tau^{6/5} - \frac{\gamma^7 f'}{14a'} \tau^{7/5} + \\
& + \left( -\frac{65\gamma^8}{4704} + \frac{250a'^2}{441\gamma^2} + \frac{3125a'^4}{588\gamma^{12}} \right) \tau^{8/5} - \\
& - \left( \frac{\gamma^9 f'}{21a'} + \frac{25a' f'}{126\gamma} \right) \tau^{9/5} + \\
& + \left( \frac{71\gamma^{10}}{113190} + \frac{112975a'^2}{271656} - \frac{18750a'^4}{3773\gamma^{10}} - \frac{234375a'^6}{7546\gamma^{20}} - \frac{\gamma^{10} f'^2}{88a'^2} - \frac{5a'\gamma'}{3\gamma} + \frac{5a''}{6} \right) \tau^2 + \\
& + \left( -\frac{173\gamma^{11} f'}{12936a'} - \frac{650\gamma a' f'}{4851} + \frac{1250a'^3 f'}{1617\gamma^9} \right) \tau^{11/5} + \\
& + \left( \frac{278263\gamma^{12}}{395531136} - \frac{117475\gamma^2 a'^2}{18540522} + \frac{85896875a'^4}{40452048\gamma^8} + \frac{448750000a'^6}{9270261\gamma^{18}} + \frac{701171875a'^8}{3090087\gamma^{28}} - \right. \\
& \quad \left. - \frac{225\gamma^2 f'^2}{1001} - \frac{12\gamma^{12} f'^2}{1001a'^2} - \frac{25\gamma a' \gamma'}{21} + \frac{625a'^3 \gamma'}{63\gamma^9} + \frac{5\gamma^2 a''}{21} - \frac{125a'^2 a''}{63\gamma^8} \right) \tau^{12/5} + \\
& + \left( -\frac{977\gamma^{13} f'}{588588a'} + \frac{285125\gamma^3 a' f'}{3531528} - \frac{93125a'^3 f'}{441441\gamma^7} - \frac{1390625a'^5 f'}{294294\gamma^{17}} - \frac{\gamma^{13} f'^3}{1144a'^3} - \right. \\
& \quad \left. - \frac{125\gamma^2 f' \gamma'}{126} + \frac{25\gamma^3 f' a''}{147a'} + \frac{25\gamma^3 f''}{882} \right) \tau^{13/5} + \\
& + \left( \frac{97897\gamma^{14}}{576816240} - \frac{191605625\gamma^4 a'^2}{4153076928} - \frac{78452875a'^4}{194675481\gamma^6} - \frac{9069078125a'^6}{259567308\gamma^{16}} - \frac{10747656250a'^8}{21630609\gamma^{26}} - \right.
\end{aligned}$$



$$\begin{aligned}
& -\frac{26869140625 a'^{10}}{14420406 \gamma^{36}} - \frac{3460 \gamma^4 f'^2}{63063} - \frac{3529 \gamma^{14} f'^2}{672672 a'^2} - \frac{21625 a'^2 f'^2}{42042 \gamma^6} - \frac{95 \gamma^3 a' \gamma'}{252} + \\
& + \frac{1250 a'^3 \gamma'}{1323 \gamma^7} - \frac{31250 a'^5 \gamma'}{441 \gamma^{17}} - \frac{2 \gamma^2 \gamma'^2}{3} - \frac{25 \gamma^4 a''}{504} - \frac{250 a'^2 a''}{1323 \gamma^6} + \\
& + \frac{6250 a'^4 a''}{441 \gamma^{16}} - \frac{\gamma^3 \gamma''}{6} \Big) \tau^{14/5} + \\
& + \Big( \frac{467 \gamma^{15} f'}{5992896 a'} + \frac{256225 \gamma^5 a' f'}{9270261} - \frac{50344625 a'^3 f'}{222486264 \gamma^5} + \frac{42781250 a'^5 f'}{9270261 \gamma^{15}} + \frac{213671875 a'^7 f'}{6180174 \gamma^{25}} - \\
& - \frac{5 \gamma^{15} f'^3}{3003 a'^3} - \frac{295 \gamma^5 f'^3}{24024 a'} - \frac{28825 \gamma^4 f' \gamma'}{38808} - \frac{55625 a'^2 f' \gamma'}{6468 \gamma^6} + \frac{170 \gamma^5 f' a''}{1029 a'} + \\
& + \frac{124625 a' f' a''}{67914 \gamma^5} - \frac{4525 \gamma^5 f''}{271656} - \frac{15625 a'^2 f''}{135828 \gamma^5} \Big) \tau^3 + O(\tau^{16/5}) .
\end{aligned} \tag{A1}$$

The expression for the coordinate  $v(\sigma, \tau)$  follows by changing  $a \rightarrow -a, u \rightarrow v$  in both sides of eq.(??).

We find for  $r(\sigma, \tau)$  near  $r = 0$ :

$$\begin{aligned}
r = & \gamma^2 \tau^{2/5} + \\
& + \left( -\frac{\gamma^4}{7} + \frac{25 a'^2}{7 \gamma^6} \right) \tau^{4/5} + \\
& - \left( \frac{4 \gamma^6}{147} + \frac{1850 a'^2}{441 \gamma^4} + \frac{2500 a'^4}{147 \gamma^{14}} \right) \tau^{6/5} + \\
& + \left( -\frac{89 \gamma^8}{11319} + \frac{40850 a'^2}{33957 \gamma^2} + \frac{908125 a'^4}{33957 \gamma^{12}} + \frac{1390625 a'^6}{11319 \gamma^{22}} + \frac{\gamma^8 f'^2}{44 a'^2} \right) \tau^{8/5} + \\
& + \left( -\frac{2735 \gamma^{10}}{1030029} + \frac{90400 a'^2}{3090087} - \frac{142715000 a'^4}{9270261 \gamma^{10}} - \frac{811718750 a'^6}{3090087 \gamma^{20}} - \frac{1068359375 a'^8}{1030029 \gamma^{30}} + \right. \\
& \quad \left. + \frac{1475 f'^2}{4004} + \frac{9 \gamma^{10} f'^2}{2002 a'^2} \right) \tau^2 + \\
& + \left( \frac{125 f' \gamma'}{63} - \frac{50 \gamma f' a''}{147 a'} - \frac{25 \gamma f''}{441} \right) \tau^{11/5} + \\
& + \left( -\frac{105436 \gamma^{12}}{108153045} - \frac{69500 \gamma^2 a'^2}{64891827} + \frac{739998250 a'^4}{194675481 \gamma^8} + \frac{45536000000 a'^6}{194675481 \gamma^{18}} + \frac{14182812500 a'^8}{4991679 \gamma^{28}} + \right. \\
& \quad \left. + \frac{205929687500 a'^{10}}{21630609 \gamma^{38}} - \frac{18230 \gamma^2 f'^2}{63063} + \frac{38 \gamma^{12} f'^2}{21021 a'^2} + \frac{8375 a'^2 f'^2}{84084 \gamma^8} + \frac{4 \gamma'^2}{3} + \frac{\gamma \gamma''}{3} \right) \tau^{12/5} + \\
& + \left( -\frac{4175 \gamma^2 f' \gamma'}{19404} + \frac{98125 a'^2 f' \gamma'}{9702 \gamma^8} - \frac{40 \gamma^3 f' a''}{1029 a'} - \frac{83375 a' f' a''}{33957 \gamma^7} + \frac{11125 \gamma^3 f''}{135828} + \right. \\
& \quad \left. + \frac{29375 a'^2 f''}{67914 \gamma^7} \right) \tau^{13/5} + O(\tau^{14/5}) .
\end{aligned} \tag{A2}$$

We get for the angular coordinate  $\phi(\sigma, \tau)$ ,

$$\begin{aligned}
\phi = & f + 2\gamma\tau^{1/5} + \\
& + \left( \frac{4\gamma^3}{21} - \frac{100a'^2}{21\gamma^7} \right) \tau^{3/5} + \\
& + \left( \frac{34\gamma^5}{735} + \frac{940a'^2}{441\gamma^5} + \frac{4250a'^4}{147\gamma^{15}} \right) \tau + \\
& + \left( \frac{164\gamma^7}{11319} - \frac{2600a'^2}{33957\gamma^3} - \frac{1037500a'^4}{33957\gamma^{13}} - \frac{2562500a'^6}{11319\gamma^{23}} - \frac{\gamma^7 f'^2}{77a'^2} \right) \tau^{7/5} + \\
& + \left( \frac{2258\gamma^9}{441441} - \frac{25700a'^2}{1324323\gamma} + \frac{47665000a'^4}{3972969\gamma^{11}} + \frac{507625000a'^6}{1324323\gamma^{21}} + \frac{882031250a'^8}{441441\gamma^{31}} - \frac{500f'^2}{9009\gamma} - \right. \\
& \quad \left. - \frac{19\gamma^9 f'^2}{3003a'^2} \right) \tau^{9/5} + \\
& + \left( \frac{20f'\gamma'}{63\gamma} + \frac{20f'a''}{147a'} + \frac{265f''}{882} \right) \tau^2 + \\
& + \left( \frac{29912\gamma^{11}}{15450435} - \frac{48200\gamma a'^2}{9270261} - \frac{42554000a'^4}{27810783\gamma^9} - \frac{7690000000a'^6}{27810783\gamma^{19}} - \frac{303125000a'^8}{64827\gamma^{29}} - \right. \\
& \quad \left. - \frac{58421875000a'^{10}}{3090087\gamma^{39}} + \frac{745\gamma f'^2}{63063} - \frac{64\gamma^{11} f'^2}{21021a'^2} + \frac{7250a'^2 f'^2}{21021\gamma^9} + \frac{4\gamma'^2}{3\gamma} + \frac{\gamma''}{3} \right) \tau^{11/5} + \\
& + \left( -\frac{5275\gamma f'\gamma'}{19404} - \frac{26875a'^2 f'\gamma'}{3234\gamma^9} + \frac{190\gamma^2 f'a''}{3087a'} + \frac{235250a' f'a''}{101871\gamma^8} - \frac{325\gamma^2 f''}{45276} - \right. \\
& \quad \left. - \frac{131875a'^2 f''}{203742\gamma^8} \right) \tau^{12/5} + O(\tau^{13/5}) .
\end{aligned}
\tag{A3}$$

We found analogous formulas for the general (non-equatorial) case with the help of Mathematica.

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